Computation of central values of L-functions

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The Problem

Let $f(z) = \sum_{n=1}^{\infty} a_n e(nz)$ be a new form of weight $2k$,
$(e(z) := e^{2\pi i z})$
D a fundamental discriminant,
Define

$$L(f, D, s) := \sum_{n=1}^{\infty} \frac{a_n}{n^s} \left(\frac{D}{n} \right).$$

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Possible method: use approximate function equation–estimation by a finite sum.
There is more efficient method in this case.

The method discussed here is through Shimura correspondence.
$f(z) \leftrightarrow g(z) = \sum_{n=1}^{\infty} c(n)e(nz)$ a weight $k + \frac{1}{2}$ cusp form.

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The question becomes:

- explicate the relation.
- compute $c(n)$.

Kohnen-Zagier Formula

Explicit relation between $L-$value and Fourier coefficient
Assume f is of level N odd and square free. Then there is a unique (up to multiple) g of level $4N$, in the Kohnen space (meaning $c(n) = 0$ if $(-1)^k n \equiv 2, 3 \bmod 4)$, satisfying $(p \nmid N)$ $T_{p^2} g/g = (\frac{(-1)^k}{p}) T_p f/f$.
Moreover when $(-1)^k D > 0$ and $(\frac{D}{p}) = w_p = \pm 1$ the eigenvalue for Atkin-Lerner involution:

$$\kappa \frac{|c(|D|)|^2}{\langle g, g\rangle} = \frac{L(f, D, k)}{\langle f, f\rangle} |D|^{k-1/2} \frac{(k-1)!}{\pi^k}.$$

where $\kappa = 2^{-\nu(N)}$, $\nu(N)$ being number of prime factors of N.
Remark: When $k = 1$, $N = p$ odd, the formula gives a relation for all $D < 0$ such that $(p, D) = 1$.

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Construction of $\Theta_1[I_i]$:
R_i left order of I_i: $\{b : bI_i \subset I_i\}$.
$S_i^0 = \{b \in Z + 2R_i : \mathrm{Tr}\, b = 0\}$
$\Theta_1[I_i] := \frac{1}{2} \sum_{b \in S_i^0} e(\mathcal{N} b\, z)$.
These are weight $3/2$ forms of level 4p. The Fourier coefficients of theta series are easy to compute. (They are integers).

Jacquet-Langlands correspondence: $f \mapsto e_f$ a function on $B^*(\mathbb{Q})\backslash B^*(\mathbb{A}_{\mathbb{Q}})/B^*_\infty R^*(\mathbb{Q}_f)$.
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Gross's construction computes $L(f,D,1)$ when $L(f,1,1) \neq 0$ and $D < 0$ with $(D,p) = 1$.

Explicate Jacquet-Langlands

Brandt matrices:
Let $M_{ij} = I_i I_j^{-1} = \{\sum a_k b_k | a_k \in I_i, b_k \in I_j^{-1}\}$. Then M_{ij} is a right ideal of R_j whose left order is R_i.
Let $\mathcal{N} M_{ij}$ be the positive greatest common divisor of $\{\mathcal{N} b | b \in M_{ij}\}$.
Define theta series ($q := e(z)$)

$$f_{ij}(z) = \frac{1}{2w_j} \sum_{b \in M_{ij}} q^{\mathcal{N} b / \mathcal{N} M_{ij}} = \frac{1}{2} \sum_{m \geq 0} B_{ij}(m) q^n.$$

Here $2w_j$ is number of units in R_j^*.

Then the Fourier coefficients $B_{ij}(m)$ give the entries of the Brandt matrix of degree m:

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Remark: can write f as a linear combination of theta series using this procedure.

Theta correspondence

Gross's construction, another description:
$f \stackrel{J-L}{\rightarrow} e_f$ is a theta correspondence between $GL_2 = GSp_2$ and $B^* \times B^* = GO(4)$ (Shimizu correspondence)
$e_f \mapsto \Theta(e_f)$ is a theta correspondence between $PB^* = O(3)$ and $\overline{SL}_2$. (Waldspurger)

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ψ: nontrivial character of $\mathbb{A}_{\mathbb{Q}}/\mathbb{Q}$

ω_ψ: Weil representation of $PB^* \times \overline{SL}_2$ acting on $\mathcal{S}(B^0)$. (B^0 the set of elements in B with trace 0).

For $\phi \in \mathcal{S}(B^0)$, $b \in PB^*$, $\sigma \in \overline{SL}_2$:

$$\theta(b, \sigma; \phi, \psi) = \sum_{x \in B^0(\mathbb{Q})} \omega_\psi(b, \sigma)\phi(x).$$

Then for some choice of ϕ

$$\Theta(e_f) = \Theta(e_f, \phi, \psi) = \int_{PB^*(\mathbb{Q})\backslash PB^*(\mathbb{A}_{\mathbb{Q}})} e_f(b)\theta(b, \ ; \phi, \psi)db.$$

Choice of ϕ: $\phi = \otimes\phi_v$ as follows

• At $v \neq 2, \infty$, ϕ_v is the characteristic function of $B^0(\mathbb{Q}_v) \cap R_v$.

• At $v = \infty$, $\phi_\infty(b) = e^{-\pi \mathcal{N} b}$.

• At $v = 2$, ϕ_2 is the characteristic function of $(1 + 2 * GL_2(\mathbb{Z}_2)) \cap B^0(\mathbb{Q}_2)$.

Waldspurger: If $\pi \leftrightarrow f$, then $\theta(JL(\pi), \psi) \neq 0$ if and only if $L(f, 1, 1) \neq 0$.

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Goal: generalize this to cases $D > 0$ or $p|D$ or $L(f, 1, 1) = 0$.
Remark: there is also a theta correspondence $\theta(\pi, \psi)$ from PGL_2 to $\overline{SL}_2$; $\theta(JL(\pi), \psi) \neq \theta(\pi, \psi)$.

Generalization of Kohnen-Zagier, (with M.Baruch)
Given π of PGL_2, let $\tilde{\pi} = \theta(\pi, \psi)$ of $\overline{SL}_2$. For $\tilde{\varphi}$ in $\tilde{\pi}$, its $\psi-$th Whittaker (Fourier) coefficient satisfies

$$|W^{\psi}(\tilde{\varphi})|^2 / \langle \tilde{\varphi}, \tilde{\varphi} \rangle \sim L(\pi, \frac{1}{2}) / L(\pi, 1, sym^2).$$

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Remark: Similar equality is expected to hold for $\tilde{\pi}$ cuspidal representation of $\overline{Sp}_{2n}$.
Apply the identity to $\tilde{\pi}^D := \theta(\pi \otimes \chi_D, \psi^D)$, $(\psi^D(x) = \psi(x/D))$, we get for $\tilde{\varphi} \in \tilde{\pi}^D$

$$|W^{\psi^D}(\tilde{\varphi})|^2 \sim L(\pi \otimes \chi_D, \frac{1}{2}) / L(\pi, 1, sym^2).$$

For $\pi \leftrightarrow f$ in our situation, Waldspurger shows:
When $D < 0$ and $(\frac{D}{p}) = w_p = \pm 1$, $\theta(\pi \otimes \chi_D, \psi^D)$ is a certain representation $\tilde{\pi}$.
$\tilde{\pi} = \theta(JL(\pi), \psi)$ when $L(f, 1, 1) \neq 0$.

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$\tilde{\pi} = \theta(JL(\pi), \psi)$ when $L(f, 1, 1) \neq 0$.
When $D > 0$ and $(\frac{D}{p}) = -w_p$, $\theta(\pi \otimes \chi_D, \psi^D)$ is another representation $\tilde{\pi}'$ of $\overline{SL_2}$,
when $L(f, 1, 1) \neq 0$, $\tilde{\pi}' = \theta(\pi, \psi)$.

Explicit formula, case $D > 0$

Fix an odd character of $(Z/p)^*$ and extend it uniquely to an even character χ of $(Z/4p)^*$. There is a unique $g'(z)$ in $S_{3/2}(4p^2, \chi)$, with $T_{l^2}g'/g' = T_l(f)/f$, $(l \neq p)$ and in Kohnen space ($c'(n) = 0$ if $(-1)^{k+1}n \equiv 2, 3 \mod 4$). Moreover when $D > 0$ and $(\frac{D}{p}) = -w_p$ we have ($k = 1$)

$$\kappa \frac{|c(|D|)|^2}{\langle g, g \rangle} = \frac{L(f, D, k)}{\langle f, f \rangle} |D|^{k-1/2} \frac{(k-1)!}{\pi^k}.$$

with $\kappa = \frac{p+1}{2p}$.

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Remark: the local component at p of $\tilde{\pi}'$ is a supercuspidal representation (the odd Weil representation), thus $g'(z)$ has a larger level. (the local component at p of $\tilde{\pi}$ is a special representation.)

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Remark: the local component at p of $\tilde{\pi}'$ is a supercuspidal representation (the odd Weil representation), thus $g'(z)$ has a larger level. (the local component at p of $\tilde{\pi}$ is a special representation.)
Remark: $p|D$ case, use $g(z)$, change κ from $1/2$ to 1.

Construction of $g(z)$ and $g'(z)$:
(with Tornaria and Rodriguez-Villegas)
Both $\tilde{\pi}$ and $\tilde{\pi}'$ has the form $\theta(JL(\pi) \otimes \chi_D, \psi^D)$ for some D.
We pick a l fundamental discriminant such that $\pm l$ is a prime, so that $L(\pi \otimes \chi_l, \frac{1}{2}) \neq 0$, then $\theta(JL(\pi) \otimes \chi_l, \psi^l) \neq 0$ and is equal to $\tilde{\pi}$ when $l > 0$, or $\tilde{\pi}'$ when $l < 0$.
Let $\varphi = \Theta(e_f(\chi_l \circ \mathcal{N}), \phi, \psi^l)$ for a suitable ϕ.

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Let $\varphi = \Theta(e_f(\chi_l \circ \mathcal{N}), \phi, \psi^l)$ for a suitable ϕ.
At $v = p$, $JL(\pi)_v \otimes \chi_l$ is trivial when $l > 0$ and a nontrivial one-dimensional representation when $l < 0$ (with our assumption $L(\pi \otimes \chi_l, \frac{1}{2}) \neq 0$).

Choice of ϕ: $\phi = \otimes \phi_v$ as follows

• At $v \neq 2, \pm l, p, \infty$, ϕ_v is the characteristic function of $B^0(\mathbb{Q}_v) \cap R_v$.

• At $v = \infty$, $\phi_\infty(b) = e^{-\pi \mathcal{N} b}$.

• At $v = 2$, ϕ_2 is the characteristic function of $(1 + 2 * GL_2(\mathbb{Z}_2)) \cap B^0(\mathbb{Q}_2)$.

• At $v = \pm l$, $\phi_l(b) = 0$ unless $b = h^{-1} \begin{pmatrix} & l \\ 1 & \end{pmatrix} h$ with h is in $GL_2(\mathbb{Z}_l)$, where $\phi_l(b) := \chi_l(\det h)$.

- At $v = p$, $B^0(\mathbb{Q}_p) = \{\begin{pmatrix} x\tau & py \\ \bar{y} & -x\tau \end{pmatrix}\}$ where $x \in \mathbb{Q}_p$,

$y \in \mathbb{Q}_p(\tau)$ the unramified quadratic extension of $\mathbb{Q}_p$.
Let ϕ_p be the characteristic function of $B^0(\mathbb{Q}_p) \cap R_p$ to get $g(z)$.
To get $g'(z)$, let $\phi_p(b) = \chi(x)$ when x and y are integral, and 0 otherwise. χ is an odd character of $(\mathbb{Z}/p)^*$.

Generalization of Gross's construction/formula

Let

$$\Theta_l[I_i] := \frac{1}{2} \sum_{b \in S_i^0} \omega_{i,l}(b)\omega_{i,p}(b)e(\mathcal{N} b\, z/l).$$

(The choice of ϕ_p and ϕ_l results in weight functions $\omega_{i,l}(b)$ and $\omega_{i,p}(b)$.)
Then $g(z)$ or $g'(z) = \sum_i \Theta_l[I_i]e_f[I_i]$.

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(The choice of ϕ_p and ϕ_l results in weight functions $\omega_{i,l}(b)$ and $\omega_{i,p}(b)$.)
Then $g(z)$ or $g'(z) = \sum_i \Theta_l[I_i]e_f[I_i]$.
Moreover we have (when $lD < 0$):

$$L(f, l, 1)L(f, D, 1) = \frac{\langle f, f\rangle}{\langle e_f, e_f\rangle} \frac{|c(|D|)|^2}{\sqrt{|Dl|}}\kappa$$

where $\kappa = 2$ when $p|D$ and $\kappa = 1$ otherwise.
This implies the construction gives nonzero forms.

Proof of the identity:

1. Theta correspondence between PB^* and $\overline{SL_2}$: if $\tilde{\varphi} = \Theta(e_f, \phi, \psi)$, then $W^D(\tilde{\varphi}) = P_\xi(\phi *_\xi e_f)$ where P_ξ is a toric period:

Let $T_\xi \subset PB^*$ be the centralizer of $\xi \in B^0$ with $N(\xi) = -D$.

$$P_\xi(\varphi) = \int_{T_\xi(\mathbb{Q})\backslash T_\xi(\mathbb{A}_\mathbb{Q})} \varphi(h)\, dh.$$

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$$P_\xi(\varphi) = \int_{T_\xi(\mathbb{Q}) \backslash T_\xi(\mathbb{A}_\mathbb{Q})} \varphi(h)\, dh.$$

2. Waldspurger's formula for toric period:
$|P_\xi(\phi *_\xi e_f)|^2 \sim |P_\xi(e_f)|^2 \sim L(\pi \otimes \chi_D, 1)L(\pi, 1)/L(\pi, Ad, 1)$.
Local calculations gives the identity.

Generalizations:

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3. Hilbert modular form case. Example of e_f is constructed by Dembelé.